Yang-Mills theory as an Abelian theory without gauge fixing

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Abstract

A general procedure to reveal an Abelian structure of Yang-Mills theories by means of a (nonlocal) change of variables, rather than by gauge fixing, in the space of connections is proposed. The Abelian gauge group is isomorphic to the maximal Abelian subgroup of the Yang-Mills gauge group, but not its subgroup. A Maxwell field of the Abelian theory contains topological degrees of freedom of original Yang-Mills fields which generate monopole-like and flux-like defects upon an Abelian projection. 't Hooft's conjecture that "monopole" dynamics is projection independent is proved for a special class of Abelian projections. A partial duality and a dynamical regime in which the theory may have massive excitations being knot-like solitons are discussed.

1. General remarks. One of the physical scenarios of the color confinement is based on the idea that the vacuum state of quantum Yang-Mills theory is realized by a condensate of monopole-antimonopole pairs [1]. In such a vacuum the field between two colored sources would be squeezed into a tube whose energy is proportional to its length. The picture is dual to the magnetic monopole confinement in a superconductor of the second kind. Monopoles as classical solutions with finite energy are absent in a pure Yang-Mills theory. To realize the dual scenario of the confinement, 't Hooft proposed an Abelian projection where the gauge group is broken by a suitable gauge condition to its maximal Abelian subgroup [2]. Since the topology of the SU(N) manifold and that of its maximal Abelian subgroup $[\mathrm{U}(1)]^{N-1}$ are different, any such gauge is singular, meaning that a gauge group element which transforms a generic SU(N) connection onto the gauge fixing surface is not regular everywhere in spacetime. The singularities may form worldlines that are usually interpreted as worldlines of magnetic monopoles (whose charges are defined with respect to the unbroken Abelian subgroup). As a result the original Yang-Mills theory turns into electrodynamics with magnetic monopoles. Recent numerical simulations show that the monopole degrees of freedom in the Abelian projection can indeed form a condensate responsible for the confinement [3].

Although the numerical results look rather appealing and stimulating, they still do not provide us with an understanding of the confinement mechanism and a nonperturbative spectrum in Yang-Mills theory. In particular, monopoles seem to emerge as an *artifact* of gauge fixing. The Abelian group appears as a *subgroup* of the full Yang-Mills gauge group. However one can easily construct colored states which are singlets with respect to the unbroken (maximal) Abelian subgroup, and, hence, they would not be confined even if the "monopoles" condense. A choice of the gauge may be convenient in practical

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computations. However, no physical phenomenon can depend on it. This suggests that in Yang-Mills theory there seems to be a new mechanism of confinement at work which has yet to be understood, and the reason of why Abelian projections work so well in the lattice theory must be explained in a gauge independent way. A first and necessary step in this direction is to reveal an Abelian structure of Yang-Mills theory without any gauge fixing.

In this letter, a Yang-Mills theory is reformulated as an Abelian gauge theory via a (nonlocal) change of variables in the space of connections, rather than via a gauge fixing (or an Abelian projection). In particular, it turns out to be possible to construct the field variables in the Abelian theory so that they are invariant under the original non-Abelian gauge transformations. Therefore an effective Abelian structure is inherent to the Yang-Mills theory and gauge independent. An Abelian vector potential carries some topological degrees of freedom of the original Yang-Mills connection which generate monopole-like and flux-like defects upon an Abelian projection (in which the gauge group is broken to its maximal Abelian subgroup by a *gauge fixing* [2]). For a rather wide class of Abelian projections, which have a characteristic property that topological defects occur in the Abelian components of projected connections, we offer theoretical arguments to prove 't Hooft's conjecture that dynamics of "monopoles" does not depend on the choice of a projection, i.e., it is gauge independent.

A generalization of the parameterization of the Yang-Mills connection proposed by Faddeev and Niemi [4] is considered as a special example. While revealing a partial duality in Yang-Mills theory, it has an important advantage that it is a *genuine* change of variables in the functional integral. Therefore it provides a description of the *off-shell* dynamics of physical degrees of freedom which is compatible with the Gauss law. Following the Wilsonian arguments of [4], we discuss the partial duality in the theory and a dynamical regime in which the topological degrees of freedom may form massive excitations being knot-like solitons.

2. Gauge group SU(2). Let $\boldsymbol{A}_\mu$ be an SU(2) connection. Consider the following parameterization of the connection

$$\boldsymbol{A}_\mu = \boldsymbol{\alpha}_\mu + \boldsymbol{n} C_\mu + \boldsymbol{W}_\mu \ , \quad \boldsymbol{\alpha}_\mu = g^{-1}\partial_\mu \boldsymbol{n} \times \boldsymbol{n} \ , \quad \boldsymbol{W}_\mu \cdot \boldsymbol{n} = 0 \ , \tag{1}$$

where g is a coupling constant, $\boldsymbol{\alpha}_\mu$ is a connection introduced by Cho [5], $\boldsymbol{n}$ is a unit isotopic vector, $\boldsymbol{n}^2 = 1$. The dot and cross stand, respectively, for the dot and cross products in the isotopic space whose elements are denoted by boldface letters. Relation (1) is not yet a genuine *change* of variables in the affine space of connections. Two more conditions have to be imposed on $\boldsymbol{W}_\mu$ in order for (1) to be a change of variables. We may set in general

$$\boldsymbol{\chi}(\boldsymbol{W}, \boldsymbol{n}, C) = 0 \ , \quad \boldsymbol{\chi} \cdot \boldsymbol{n} \equiv 0 \ . \tag{2}$$

The function $\boldsymbol{\chi}$ can be chosen so that a solution of Eq. (2) determines a *local* and *explicit* parameterization of eight components in $\boldsymbol{W}_\mu$ by six functional variables (see section 5), thus leading to a generalization of the parameterization given in [4]. We will also show that some $\boldsymbol{\chi}$'s, for which Eq. (2) admits *nonlocal* parameterizations of the SU(2) connection, can naturally be associated with 't Hooft's Abelian projections where an Abelian vector potential contains magnetic monopoles described by the field $\boldsymbol{n}$.

Before specifying $\boldsymbol{\chi}$ let us first analyze the gauge transformation law of the new variables. An infinitesimal gauge transformation of the SU(2) connection reads

$$\delta \boldsymbol{A}_\mu = g^{-1}\nabla_\mu(\boldsymbol{A})\boldsymbol{\varphi} = g^{-1}\left[\partial_\mu \boldsymbol{\varphi} + g\boldsymbol{A}_\mu \times \boldsymbol{\varphi}\right] \ . \tag{3}$$

From (1) we infer

$$C_\mu = \boldsymbol{n}\cdot\boldsymbol{A}_\mu \ , \qquad \boldsymbol{W}_\mu = g^{-1}\boldsymbol{n}\times\nabla(\boldsymbol{A})\boldsymbol{n} \ . \tag{4}$$

Substituting these relations into (2) and solving them for $\boldsymbol{n}$ (two equations for two independent variables in $\boldsymbol{n}$), we find $\boldsymbol{n} = \boldsymbol{n}(\boldsymbol{A})$. The latter together with (4) specifies the inverse change of variables. Let $\delta\boldsymbol{n}$ be an infinitesimal gauge transformation of $\boldsymbol{n}$. Then from (4) and (3) it follows that

$$\begin{aligned} \delta C_\mu &= \boldsymbol{A}_\mu \cdot (\delta\boldsymbol{n} - \boldsymbol{n}\times\boldsymbol{\varphi}) + g^{-1}\boldsymbol{n}\cdot\partial_\mu\boldsymbol{\varphi} \ , & (5)\\ \delta\boldsymbol{W}_\mu &= \boldsymbol{W}\times\boldsymbol{\varphi} - \boldsymbol{n}[\boldsymbol{W}_\mu\cdot(\delta\boldsymbol{n} - \boldsymbol{n}\times\boldsymbol{\varphi})] + g^{-1}\boldsymbol{n}\times\partial_\mu(\delta\boldsymbol{n} - \boldsymbol{n}\times\boldsymbol{\varphi}) \ , & (6) \end{aligned}$$

where we have used that $\boldsymbol{n}\cdot\delta\boldsymbol{n} = 0$. An explicit form of $\delta\boldsymbol{n}$ can be found from the equation $\delta\boldsymbol{\chi}(\boldsymbol{n},\boldsymbol{A}) = 0$ (taken on the surface $\boldsymbol{\chi} = 0$) where $\boldsymbol{\chi}(\boldsymbol{n},\boldsymbol{A})$ is obtained by a substitution of (4) into $\boldsymbol{\chi}(\boldsymbol{W}, C, \boldsymbol{n})$. We emphasize that $\delta\boldsymbol{n}$ is determined by the choice of $\boldsymbol{\chi}$ and so are δC_μ and $\delta\boldsymbol{W}_\mu$.

Let us introduce a local orthonormal basis in the isotopic space $\boldsymbol{n}$, $\boldsymbol{n}_r$ and $\boldsymbol{n}_r^*$: $\boldsymbol{n}_r\cdot\boldsymbol{n} = 0$, $\boldsymbol{n}_r^2 = 0$ and $\boldsymbol{n}_r\cdot\boldsymbol{n}_r^* = 1$. We also have $\boldsymbol{n}\times\boldsymbol{n}_r = i\boldsymbol{n}_r$ and $\boldsymbol{n}_r\times\boldsymbol{n}_r^* = i\boldsymbol{n}$. With $\boldsymbol{n}$ fixed, the basis is determined modulo *local* transformations

$$\boldsymbol{n}_r \to e^{i\xi}\boldsymbol{n}_r \ . \tag{7}$$

It should be noted that this gauge freedom is *not* associated with the gauge group of the Yang-Mills theory because the new variables remain unchanged under (7). The local basis may not exist globally and the field $\boldsymbol{n}_r$ may have singularities. The reason is as follows. At the spatial infinity, the connection must be a pure gauge. Therefore $\boldsymbol{n}$ becomes a constant as $|\boldsymbol{x}|$ approaches infinity, say, $\boldsymbol{n}_0 = (0,0,1)$. The field $\boldsymbol{n}$ is a map of the three-sphere S^3, being the compactified space, to the target two-sphere S^2 in the isotopic space. The homotopy group $\pi_3(S^2)\sim Z$ is not trivial. Integers from Z are given by the Hopf invariant. If one attempts to transform $\boldsymbol{n}$ to $\boldsymbol{n}_0$ everywhere in space by rotation, the rotation matrix will be ill-defined on some closed and, in general, knotted contours. Another type of singularities is associated with the homotopy group $\pi_2(S^2)\sim Z$ when the field $\boldsymbol{n}$ is restricted on some S^2 being a subspace of S^3. We will show that in the new variables (1), the Yang-Mills theory looks like an Abelian theory in which a Maxwell potential contains magnetic monopoles and fluxes associated with the nontriviality of $\pi_2(S^2)$ and $\pi_3(S^2)$, respectively.

Consider the decomposition

$$\boldsymbol{W}_\mu = W_\mu^*\boldsymbol{n}_r + W_\mu\boldsymbol{n}_r^* \ . \tag{8}$$

The fields strength is, by definition, $\boldsymbol{F}_{\mu\nu}(\boldsymbol{A}) = \partial_\mu \boldsymbol{A}_\nu - \partial_\nu \boldsymbol{A}_\mu + g\boldsymbol{A}_\mu \times \boldsymbol{A}_\nu$. In particular,

$$\begin{aligned} \boldsymbol{F}_{\mu\nu}(\boldsymbol{\alpha} + \boldsymbol{n}C) &= \boldsymbol{n}(C_{\mu\nu} - H_{\mu\nu}) \ , \quad C_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu \ , & (9) \\ H_{\mu\nu} &= g^{-1}\boldsymbol{n} \cdot (\partial_\mu \boldsymbol{n} \times \partial_\nu \boldsymbol{n}) = \partial_\mu H_\nu - \partial_\nu H_\mu - H^{(st)}_{\mu\nu} \ , & (10) \end{aligned}$$

where $H_\mu = ig^{-1}\boldsymbol{n}^*_r \cdot \partial_\mu \boldsymbol{n}_r = H^*_\mu$ and $H^{(st)}_{\mu\nu} = ig^{-1}\boldsymbol{n}^*_r \cdot [\partial_\mu, \partial_\nu]\boldsymbol{n}_r = H^{(st)*}_{\mu\nu}$ which is the field strength of Dirac strings associated with the singularities of the local basis. For example, the Wu-Yang monopole configuration is determined by $\boldsymbol{n} = \boldsymbol{x}/r$, $r^2 = \boldsymbol{x}^2$ and $C_\mu = W_\mu = 0$. The Dirac string is extended along the negative part of the z-axis (if $\boldsymbol{x} = (x, y, z)$). It is also not difficult to give an example of $\boldsymbol{n}$ for which Dirac strings would form closed linked and/or knotted contours (see, e.g., [6]). Thus, the vector potential H_μ describes possible monopole-like and closed-flux-like degrees of freedom to which we refer as to topological degrees of freedom in the Yang-Mills theory. After a modest computation we obtain

$$\boldsymbol{F}^2_{\mu\nu} = \Big[C_{\mu\nu} - H_{\mu\nu} + ig(W^*_\mu W_\nu - W^*_\nu W_\mu)\Big]^2 + |D_\mu W_\nu - D_\nu W_\mu|^2 \ , \tag{11}$$

where $D_\mu W_\nu = \partial_\mu W_\nu - igA_\mu W_\nu$ is the U(1) covariant derivative, $A_\mu = C_\mu - H_\mu$. The Abelian gauge transformations have the form

$$A_\mu \to A_\mu + g^{-1}\partial_\mu \xi \ , \quad W_\mu = e^{i\xi} W_\mu \ . \tag{12}$$

The transformations (12) can obviously be generated by (7), and therefore they are not from the original SU(2) gauge group. In contrast to the Abelian gauge transformations (12), the SU(2) transformations depend on a concrete parameterization of $\boldsymbol{W}_\mu$. Because of topological defects associated with the nontriviality of $\pi_2(S^2)$, the Bianchi identity for the Abelian strength tensor $F_{\mu\nu} = C_{\mu\nu} - H_{\mu\nu}$ is violated. Let ${}^\star F_{\mu\nu}$ be the dual tensor. Then one can define a conservative current

$$J_\mu = \partial_\nu {}^\star F_{\mu\nu} \ , \quad \partial_\mu J_\mu = 0 \ . \tag{13}$$

The conservation of the topological current J_μ indicates the existence of the U(1) (magnetic) symmetry on the classical level in the theory (11).

3. Abelian projections. An Abelian projection is introduced by imposing a gauge condition on $\boldsymbol{A}_\mu$ that breaks the gauge group SU(2) to its (maximal) Abelian subgroup U(1). As has already been pointed out in section 1, a gauge group element which transforms a generic connection to the gauge fixing surface in the space of connections is not regular everywhere in spacetime. The transformed (or projected) connections contain topological defects (singularities). Consider a special class of Abelian projections with the characteristic property that topological defects appear only in the Abelian components of the projected potentials. Let us describe this class in the new variables (1). Had the gauge transformations of the field $\boldsymbol{n}$ been just isotopic rotations,

$$\delta \boldsymbol{n} = \boldsymbol{n} \times \boldsymbol{\varphi} \ , \tag{14}$$

then the connection $\boldsymbol{\alpha}_\mu + \boldsymbol{n}C_\mu$ would become purely Abelian upon the projection $\boldsymbol{n} \to \boldsymbol{n}_0$ for *any* choice of $\boldsymbol{\chi}$ which is *compatible* with (14). Recall that in our formulation the gauge transformation law for $\boldsymbol{n}$ depends on $\boldsymbol{\chi}$. The addition to the Abelian component C_μ resulting from $\boldsymbol{\alpha}_\mu$ upon the projection determines exactly the same topological defects as the connection H_μ in the Abelian theory (11), i.e., $\boldsymbol{\alpha}_\mu + \boldsymbol{n}C_\mu \to \boldsymbol{n}_0 A_\mu$.

Now we show that for *every* Abelian projection from the special class defined above one can construct $\boldsymbol{\chi}$ in (2) which determines a special parameterization of $\boldsymbol{W}_\mu$ such that (14) holds. Moreover the topological current (13) is *invariant* under the SU(2) gauge transformations (5), (6) and (14). Thus, the very existence of the (magnetic) symmetry (13) is not at all related to any gauge fixing in the theory. We first give an example of $\boldsymbol{\chi}$ associated with the so called maximal Abelian projection [7] which is mostly used in numerical studies of the "monopole" dynamics:

$$\boldsymbol{\chi} = \nabla_\mu(\boldsymbol{\alpha} + \boldsymbol{n}C)\boldsymbol{W}_\mu \ . \tag{15}$$

The compatibility of (15) with (14) follows from the fact [8] that under the transformations (5), (6) and (14) the isovector (15) is covariant, $\delta\boldsymbol{\chi} = \boldsymbol{\chi}\times\boldsymbol{\varphi}$, and it also fulfills the condition $\boldsymbol{\chi}\cdot\boldsymbol{n} \equiv 0$ as one can easily be convinced by a direct computation. Upon the projection $\boldsymbol{n} \to \boldsymbol{n}_0$, $\boldsymbol{\chi}$ turns into the maximal Abelian gauge condition. The Abelian part of the connection equals A_μ and contains magnetic monopoles whose charges are defined with respect to the unbroken U(1) subgroup (rotations about $\boldsymbol{n}_0$) [8]. By construction, the corresponding conservative monopole current coincides with (13).

Suppose an Abelian projection is specified by a gauge condition $\boldsymbol{\chi}(\boldsymbol{n}_0 A^{(0)}, \boldsymbol{W}^{(0)}) = 0$, where $\boldsymbol{A}_\mu = \boldsymbol{n}_0 A_\mu^{(0)} + \boldsymbol{W}_\mu^{(0)}$ and $\boldsymbol{n}_0 \cdot \boldsymbol{W}_\mu^{(0)} = 0$. We also assume that $\boldsymbol{\chi}$ is covariant (or even invariant) under the Abelian gauge transformations $\delta_a A_\mu^{(0)} = g^{-1}\partial_\mu\varphi$ and $\delta_a \boldsymbol{W}_\mu^{(0)} = \varphi\boldsymbol{W}_\mu^{(0)} \times \boldsymbol{n}_0$. This ensures that the gauge symmetry is broken to U(1). Consider the change of variables (1) in which the condition (2) is obtained by a simple replacement $\boldsymbol{n}_0 A_\mu^{(0)} \to \boldsymbol{\alpha}_\mu + \boldsymbol{n}C_\mu$ and $\boldsymbol{W}_\mu^{(0)} \to \boldsymbol{W}_\mu$ in the above gauge condition. By construction, the gauge transformation law (14) is guaranteed. All topological degrees of freedom of the Yang-Mills theory, which are singled out as magnetic monopoles upon the Abelian projection, are contained in the Abelian vector potential A_μ of the Maxwell theory (11).

Thus, in the new variables the aforementioned special class of Abelian projections is described by a single "projection" $\boldsymbol{n} \to \boldsymbol{n}_0$.

From (9) and (10) it follows that

$$\delta J_\mu = \partial_\nu {}^\star\delta F_{\mu\nu} = 0 \ , \tag{16}$$

that is, the topological current (13) is *invariant* under the SU(2) gauge transformations. Note that C_μ is not transformed by a simple gradient shift. The contribution of non-Abelian gauge transformations to $\delta C_{\mu\nu}$ is compensated in $\delta F_{\mu\nu}$ by $\delta H_{\mu\nu}$ which is generated by gauge rotations of the local basis $\delta\boldsymbol{n}_r = \boldsymbol{n}_r \times \boldsymbol{\varphi}$, $\boldsymbol{\varphi}\cdot\boldsymbol{n} = 0$. The restriction on $\boldsymbol{\varphi}$ has been imposed because $\delta\boldsymbol{n} = 0$ if $\boldsymbol{\varphi} = \boldsymbol{n}\varphi$, while $\delta C_\mu = g^{-1}\partial_\mu\varphi$. Therefore there are two groups U(1) in the theory. One is associated with the subgroup of the gauge group which preserve $C_{\mu\nu}$, $\delta C_{\mu\nu} = 0$, while the other is given by transformations (12). The Abelian potential A_μ is *invariant* under the U(1) subgroup of U(1)×U(1) which is selected by the condition $\varphi = \xi$. The charged field W_μ is also *invariant* under this U(1) subgroup.

According (6) and (8), the SU(2) gauge transformations can be regarded as local generic rotations of any *rigid* local basis in the isotopic space, $\delta \boldsymbol{n}_r = \boldsymbol{n}_r \times \boldsymbol{\varphi}$ (no restriction on $\boldsymbol{\varphi}$), provided the $\boldsymbol{\chi}$ in (2) is such that (14) holds. In this case all field variables in the Maxwell theory (11) are invariant under the SU(2) Yang-Mills gauge group.

4. 't Hooft's conjecture. In lattice simulations one is interested in an effective dynamics of the topological degrees of freedom, i.e., in an effective theory of the field $\boldsymbol{n}$ in our formulation. Recently the dual scenario of the color confinement in the lattice Yang-Mills theory has been reported to occur in several Abelian projection [9]. All the projections studied have a characteristic property that monopole-like topological defects are contained in Abelian components of projected connections. This certainly supports 't Hooft's conjecture that all Abelian projections are equivalent [2]. Can one find theoretical arguments to prove this conjecture? Here we explain how the proof can be done.

In our parameterization all the Abelian projections in question are described by one simple (singular) gauge condition $\boldsymbol{n} = \boldsymbol{n}_0$. The difference between projections is related to a *reparameterization* of $\boldsymbol{W}_\mu$. As we have a *genuine* change of variables in the functional integral, we can, in principle, integrate out $\boldsymbol{W}_\mu$, and get an effective action for $\boldsymbol{n}$ and C_μ. From a technical point of view, this procedure involves two important steps. First, one has to compute a Jacobian of the change of variables. Second, a gauge has to be fixed, otherwise the integral is divergent. The latter can be done by means of the conventional Faddeev-Popov recipe with a nonsingular gauge (e.g. a background or Lorentz gauge) before the change of variables. The first problem is solved in the following way [8]. Consider the identity $1 = \int D\boldsymbol{n}\Delta(\boldsymbol{A},\boldsymbol{n})\delta(\boldsymbol{\chi})$ where $\boldsymbol{\chi} = \boldsymbol{\chi}(\boldsymbol{A},\boldsymbol{n})$ is obtained by a substitution of (4) into $\boldsymbol{\chi}(\boldsymbol{W}, C, \boldsymbol{n})$. Clearly, $\Delta(\boldsymbol{A},\boldsymbol{n}) = \det[\delta\boldsymbol{\chi}/\delta\boldsymbol{n}]$. Next, the identity is inserted into the integral $\int D\boldsymbol{A}_\mu \exp(-S)$, with S being the Yang-Mills action (gauge fixing and Faddeev-Popov ghost terms are not written explicitly), then $\boldsymbol{A}_\mu$ is replaced by $\boldsymbol{n}C_\mu + \boldsymbol{W}_\mu$, with a *generic* $\boldsymbol{W}_\mu$ perpendicular to $\boldsymbol{n}$ so that $D\boldsymbol{A}_\mu \sim DC_\mu D\boldsymbol{W}_\mu$. Finally, one shifts the integration variables $\boldsymbol{W}_\mu \to \boldsymbol{W}_\mu + \boldsymbol{\alpha}_\mu$. As a result one arrives at the following representation

$$\mathcal{Z} \sim \int D\boldsymbol{A}_\mu \, e^{-S} \sim \int D\boldsymbol{n} DC_\mu D\boldsymbol{W}_\mu \Delta(\boldsymbol{A},\boldsymbol{n})\delta[\boldsymbol{\chi}(\boldsymbol{W}, C, \boldsymbol{n})]\, e^{-S} \, . \tag{17}$$

In the integrand of the right-hand side of Eq. (17), $\boldsymbol{A}_\mu$ must be replaced by (1). The integral over $\boldsymbol{W}_\mu$ seems to depend on the choice of $\boldsymbol{\chi}$. However, this is not always the case.

Various choices of $\boldsymbol{\chi}$ can be regarded as *gauge fixing* conditions for the gauge symmetry associated with a reparameterization of $\boldsymbol{W}_\mu$. As it stands, Eq.(1) contains 14 functions in the right-hand side, while there are only 12 components in $\boldsymbol{A}_\mu$. Therefore the gauge transformations (3) would, in general, be induced by *five*-parametric transformations of the new variables. There are two-parametric transformations of the new variables under which $\boldsymbol{A}_\mu$ remains invariant. Precisely this gauge freedom is fixed by (2) and by the corresponding delta function in (17). The key point is that the invariance of the integral over $\boldsymbol{W}_\mu$ in (17) under variations of $\boldsymbol{\chi}$ can be established just as the gauge invariance of the perturbative Faddeev-Popov path integral is proved. Since Δ is a determinant, it can be lifted up to the exponential by introducing ghosts $\boldsymbol{\eta}$ (which should not be confused with the conventional Faddeev-Popov ghosts), and $\delta(\boldsymbol{\chi})$ is replaced by $\int D\boldsymbol{f} \exp(-\boldsymbol{f}^2/2)\delta(\boldsymbol{\chi} - \boldsymbol{f})$. A change of $\boldsymbol{\chi}$ is equivalent to some BRST transformation of $\boldsymbol{\eta}$ and the new variables. When $\boldsymbol{W}_\mu$ is integrated out, the invariance of the effective action for the remaining variables under variations of $\boldsymbol{\chi}$ should be guaranteed by the invariance under the corresponding BRST transformations of $\boldsymbol{n}$ and C_μ. Now we recall that a change of $\boldsymbol{\chi}$ implies a modification of the gauge transformation law of $\boldsymbol{n}$ and C_μ. But for all Abelian projections in question $\boldsymbol{\chi}$ varies within the special class for which $\boldsymbol{n}$ transforms according to (14), that is, neither the gauge transformation of $\boldsymbol{n}$ nor C_μ depend on $\boldsymbol{W}_\mu$. Hence, the BRST transformations of $\boldsymbol{n}$ and C_μ generated by varying $\boldsymbol{\chi}$ cannot be anything, but a subset of the conventional BRST transformations associated with a gauge fixing in the original integral over $\boldsymbol{A}_\mu$. Owing to the BRST invariance of the Faddeev-Popov action, we conclude that the effective action for $\boldsymbol{n}$ and C_μ will also be invariant under the BRST transformations generated by variations of $\boldsymbol{\chi}$. In short, one can say that 't Hooft's conjecture is a simple consequence of the gauge invariance. In the new nonlocal variables, a relevance of the gauge symmetry is obvious, while in the original variables it is less evident because of singularity of gauges used in Abelian projections. The general case when (14) is not valid will be considered elsewhere.

If the dual scenario takes place, as suggested by lattice simulations, the effective action for the topological current J_μ has to be of the London type (as for superconductor). Since the Abelian theory (11) is SU(2) invariant, the U(1) symmetry associated with the conservation of J_μ can be dynamically broken *regardless* of any gauge fixing used to compute the functional integral. By means of the representation (17), where the integration over the field $\boldsymbol{n}$ provides the sum over topological configurations of Yang-Mills fields, we have circumvented a difficult problem of summing over monopole configurations in singular Abelian projection gauges.

5. Partial duality. The homotopy group arguments show that the field $\boldsymbol{n}$ may also contain configurations that upon the Abelian projection $\boldsymbol{n} \to \boldsymbol{n}_0$ form closed magnetic fluxes which are linked and/or knotted. Their topological number is known as the Hopf invariant and associated with the nontriviality of $\pi_3(S^2)$ of the map $\boldsymbol{n}$. Due to a nonlocality of the Hopf invariant, there is no conservative current related to such topological defects. The magnetic fluxes cannot be observed in numerical studies by the same procedure as that used for magnetic monopoles because they do not contribute to the total magnetic field flux through any closed surface. It has been conjectured that quantum fluctuations of other degrees of freedom of Yang-Mills fields may stabilize fluxes against shrinking so that they would behave like knot solitons [4]. The dynamical regime in which fluxes exist as physical excitations is dual to some Higgs phase which is revealed via a special parameterization of the Yang-Mills connection [4].

To verify this conjecture, one needs a more general parameterization of the connection than that used in [4] in order to correctly describe an *off-shell* quantum dynamics of relevant physical degrees of freedom. The problem is to find an *explicit* and *local* parameterization of $\boldsymbol{W}_\mu$ by six functional variables, while keeping the partial duality between $\boldsymbol{n}$ and some components of $\boldsymbol{W}_\mu$. Then in the new variables the Yang-Mills theory will be a local Abelian theory (11) to which the Wilsonian arguments of [4] can be applied.

The necessary six functional variables can be unified into an antisymmetric tensor $W_{\mu\nu} = -W_{\nu\mu}$. Consider the following representation

$$\boldsymbol{W}_\mu = g\left[W_{\mu\nu} + V_{\mu\nu}(W, \boldsymbol{n})\right]\boldsymbol{\alpha}_\nu \ , \tag{18}$$

where $V_{\mu\nu}$ is a symmetric tensor which depends on $W_{\mu\nu}$ and $\boldsymbol{n}$. This is the most general form of $\boldsymbol{W}_\mu$. It should be noted that $W_{\mu\nu}$ is *dimensionless* just as the topological field $\boldsymbol{n}$, which is necessary for $W_{\mu\nu}$ to be a dual variable to $\boldsymbol{n}$. In principle, one can take a generic isotopic vector $\boldsymbol{\gamma}_\mu(\boldsymbol{n})$, perpendicular to $\boldsymbol{n}$, instead of $\boldsymbol{\alpha}_\mu$ in (18). However, by a redefinition of the symmetric and antisymmetric components of the tensor $W_{\mu\nu} + V_{\mu\nu}$, $\boldsymbol{\gamma}_\mu$ can always be replaced by $\boldsymbol{\alpha}_\mu$ because any isotopic vector perpendicular to $\boldsymbol{n}$ is a linear combination of the Lorentz components of $\boldsymbol{\alpha}_\mu$. The simplest choice $V_{\mu\nu} = 0$ would already provide us with a sought-for parameterization to develop the off-shell dynamics of the physical degrees of freedom. It implies only *one* gauge condition on a generic connection (1), while we are allowed to impose *three* without solving the Gauss law. Indeed, if $V_{\mu\nu} = 0$, $\boldsymbol{W}_\mu$ satisfies *three* (not *two* as required by (2)) equations

$$\boldsymbol{W}_\mu \otimes \boldsymbol{\alpha}_\mu + \boldsymbol{\alpha}_\mu \otimes \boldsymbol{W}_\mu = 0 \ . \tag{19}$$

The tensor product contains three independent components because both $\boldsymbol{W}_\mu$ and $\boldsymbol{\alpha}_\mu$ are perpendicular to $\boldsymbol{n}$. Therefore there is one constraint on the components of $W_{\mu\nu}$: $W_{\mu\nu}H_{\mu\nu} = 0$. Since for the functional integral one needs a *change* of variables, the latter restriction on $W_{\mu\nu}$ can be relaxed to achieve this goal if, for example, we set

$$\boldsymbol{W}_\mu = gW_{\mu\nu}\boldsymbol{\alpha}_\nu + g\rho\boldsymbol{\alpha}_\mu \ , \tag{20}$$

where $\rho = \rho(W, \boldsymbol{n}) \sim W_{\mu\nu}H_{\mu\nu}$ is determined by (19). The field ρ in (20) is, in general, specified modulo a factor which may depend on $\boldsymbol{n}$. For instance, the condition (19) can be modified by multiplying each of the two terms in the tensor product by coefficients depending on $\boldsymbol{n}$.

Thanks to the gauge invariance of the Yang-Mills action, a particular choice of ρ should not be relevant for the partial duality because ρ can always be removed by an appropriate gauge transformation (6). Quantum dynamics of the charged fields in the Abelian theory (11) is described by the antisymmetric field $W_{\mu\nu}$. The Jacobian of the change of variables is the determinant of the Euclidean metric $ds^2 = \int dx d\boldsymbol{A}_\mu \cdot d\boldsymbol{A}_\mu$ on the affine space of connections in the new variables (1) and (20); $d\boldsymbol{A}_\mu$ denotes a functional differential of the affine (field) coordinate $\boldsymbol{A}_\mu$. The Jacobian induces quantum corrections, associated with the curvilinearity of the new field variables, to the classical action (11). If the dynamics of the charged field $W_{\mu\nu}$ is such that the average over them yields

$$\langle(\partial_\mu W_{\nu\sigma} - \partial_\nu W_{\mu\sigma})\left(\partial_\mu W_{\nu\lambda} - \partial_\nu W_{\mu\lambda}\right)\rangle \sim m^2\delta_{\sigma\lambda} \ , \tag{21}$$

then in the large distance limit, the leading term of the gradient expansion of the effective action for the field $\boldsymbol{n}$ would contain the term $m^2\boldsymbol{\alpha}_\mu \cdot \boldsymbol{\alpha}_\mu = m^2(\partial_\mu \boldsymbol{n})^2$. Together with the tree level term proportional to $H_{\mu\nu}^2$, it forms, as follows from (11), the action of the

Faddeev model [10] which describes knot-like massive solitons. Such solitonic excitations could be good candidates for glueballs. Their stability in the effective theory depends on other terms which are contained in the gradient expansion of the effective action.

Consider the decomposition $\partial_\mu \boldsymbol{n} = b^*_\mu \boldsymbol{n}_r + b_\mu \boldsymbol{n}^*_r$. We have $H_{\mu\nu} = ig^{-1}(b^*_\mu b_\nu - b_\mu b^*_\nu)$ and $W_\mu = iW_{\mu\nu} b_\mu + i\rho b_\mu$. The dual (Higgs) phase reported in [4] may also exist in the Abelian theory (11), provided the average over the field $\boldsymbol{n}$ has the property that

$$\langle b_\mu b_\nu \rangle = 0 \ , \qquad \langle b^*_\mu b_\nu \rangle \sim M^2 \delta_{\mu\nu} \ . \tag{22}$$

In particular, the property (22) implies that $\langle H_{\mu\nu}\rangle = 0$ and $\langle H_{\mu\sigma} H_{\nu\lambda}\rangle \approx 2g^{-2}M^4(\delta_{\mu\nu}\delta_{\sigma\lambda} - \delta_{\mu\lambda}\delta_{\sigma\nu})$ (neglecting by a four-point function of the field b_μ). The effective potential for $W_{\mu\nu}$ would have "classical" minima: $\langle W_{\mu\sigma} W_{\nu\sigma}\rangle \sim \delta_{\mu\nu}$, therefore the Maxwell field acquires a mass proportional to M.

The parameterization relevant for the partial duality is given by the first term in (18). Therefore the choice of $V_{\mu\nu}$ does not seem to be important. This suggests that the property (21) should be universal relative to a choice of $V_{\mu\nu}$. The Ansatz (18) can be used to solve Eq. (2) for $V_{\mu\nu}$. In this case $V_{\mu\nu}$ may even be nonlocal (cf., e.g., (15)). It would be interesting to find arguments to prove that (21) holds for any $V_{\mu\nu} = V_{\mu\nu}(W, \boldsymbol{n})$ if it holds for at least one choice of $V_{\mu\nu}$. This amounts to the existence of the gauge $V_{\mu\nu} = 0$ for any $\boldsymbol{\chi}$ (for algebraic conditions like (19), this is the case). The nontriviality of the problem is that the gauge transformation law of the new variables depends on $\boldsymbol{\chi}$.

6. Gauge group SU(N). To extend our description of the Yang-Mills theory as an Abelian theory with topological degrees of freedom to the gauge group SU(N), we introduce the Cartan-Weyl basis in the Lie algebra [11]. Let ω_k be simple roots, $k = 1, 2, ..., N-1$ (= rank of SU(N)), and β be a positive root. It can be written in the form $\beta = \omega_k + \omega_{k+1} + \cdots + \omega_{k+j}$. All simple roots have the same norm. The angle between ω_k and $\omega_{k\pm 1}$ is $2\pi/3$, while ω_k and $\omega_{k\pm j}$, $j \geq 2$, are perpendicular. As a consequence, all roots have the same norm. For every root β two basis elements e_β and $e_{-\beta} = e^*_\beta$ are defined so that

$$[h, e_\beta] = (h, \beta) e_\beta \ , \qquad [e_\beta, e_\gamma] = N_{\beta,\gamma} e_{\beta+\gamma} \ , \qquad [e_\beta, e_{-\beta}] = \beta \ , \tag{23}$$

where h is any element from the Cartan subalgebra; and for any two elements v and w of the Lie algebra the Killing form is defined by $(v, w) = \mathrm{tr}\,(\mathrm{ad}\, v \mathrm{ad}\, w)$. The operator $\mathrm{ad}\, v$ acts on any element w as $[v, w]$. The structure constants $N_{\beta,\gamma} = -N_{-\beta,-\gamma}$ are not zero only if $\beta + \gamma$ is a root. For SU(N), $N^2_{\beta,\gamma} = (2N)^{-1}$ and relative signs can be fixed by the Jacobi identity for the basis elements. Let $h_k = h^*_k$ be an orthonormal basis with respect to the Killing form in the Cartan subalgebra. With the normalization of the structure constants as given in (23), the elements h_k, e_β and e^*_β form an orthonormal basis in the Lie algebra, $(h_k, e_\beta) = 0$, $(e_\beta, e_\gamma) = 0$ and $(e_\beta, e^*_\gamma) = \delta_{\beta\gamma}$.

Let $U = U(x)$ be a generic element of the coset $SU(N)/[U(1)]^{N-1}$. Consider a local orthonormal basis $n_k = U^\dagger h_k U$, $n_\beta = U^\dagger e_\beta U$. The commutation relations (23) hold for the local basis too. For any element v one can prove the identity

$$v = N[n_k, [n_k, v]] + n_k (n_k, v) \ . \tag{24}$$

A proof is based on a straightforward computation of the double commutator in (24) in the Cartan-Weyl basis and the fact that all roots of SU(N) have the same norm which is $(\beta,\beta)=1/N$ relative to the Killing form. A *change* of variables in the affine space of SU(N) connections reads

$$A_\mu=\alpha_\mu+n_kC^k_\mu+W_\mu\ ,\quad \alpha_\mu=ig^{-1}N[\partial_\mu n_k,n_k]\ ,\quad (W_\mu,n_k)=0\ , \tag{25}$$

where W_μ is subject to N^2-N conditions $\chi(W,C^k,n_k)=0$, $(\chi,n_k)\equiv 0$ Thus, in four dimensional spacetime, $4(N^2-1)$ independent components of A_μ are now represented by $N^2-N=\dim SU(N)/[U(1)^{N-1}]$ independent components of n_k, by $4(N-1)$ components of C^k_μ and by $3(N^2-N)$ components of W_μ. The two first terms in (25) are constructed so that the corresponding field strength is purely Abelian in the local basis

$$F_{\mu\nu}(\alpha+C)=n_k\left(C^k_{\mu\nu}-H^k_{\mu\nu}\right)\equiv n_kF^k_{\mu\nu}\ ,\quad H^k_{\mu\nu}=ig^{-1}N\left(n_k,[\partial_\mu n_j,\partial_\nu n_j]\right)\ , \tag{26}$$

and $C^k_{\mu\nu}=\partial_\mu C^k_\nu-\partial_\nu C^k_\mu$. Relation (26) is obtained from the definition $F_{\mu\nu}(A)=\partial_\mu A_\nu-\partial_\nu A_\mu+ig[A_\mu,A_\nu]$ by a successive use of the Jacobi identity and (24).

The homotopy groups $\pi_2(G/H)$ and $\pi_3(G/H)$, where $G=SU(N)$ and $H=[U(1)]^{N-1}$, of the map n_k are nontrivial. Therefore n_k carry topological (physical) degrees of freedom of Yang-Mills fields. It is not hard to establish the identity

$$ig^{-1}\partial_\mu U^\dagger U=\alpha_\mu+n_kH^k_\mu\ ,\qquad H^k_{\mu\nu}=\partial_\mu H^k_\nu-\partial_\nu H^k_\mu-H^{k(st)}_{\mu\nu}\ , \tag{27}$$

where the group element U specifies an orientation of the local basis with respect to the Cartan-Weyl basis, and $H^{k(st)}_{\mu\nu}=ig^{-1}(n_k,[\partial_\mu,\partial_\nu]U^\dagger U)$ is the field of Dirac strings. The Abelian strength tensor $F^k_{\mu\nu}$ does not satisfy the Bianchi identity if the monopole-like defects associated with a nontriviality of $\pi_2(G/H)$ are present. The conservative topological current is a simple multi-component generalization of (13): $J^k_\mu=\partial_\nu\,{}^\star F^k_{\mu\nu}$, $\partial_\mu J^k_\mu=0$. There is a (magnetic) symmetry group $[U(1)]^{N-1}$ responsible for its conservation.

To reformulate the Yang-Mills theory as an Abelian theory without gauge fixing, we introduce the decomposition $W_\mu=W^{\beta *}_\mu n_\beta+W^\beta_\mu n^*_\beta$ and set $A^k_\mu=C^k_\mu-H^k_\mu$. The Lagrangian density in the new variables assumes the form

$$\begin{aligned}F^2_{\mu\nu}&=\left\{C_{\mu\nu}-H_{\mu\nu}+ig(h_k,\beta)\left[W^{\beta*}_\mu W^\beta_\nu-W^{\beta*}_\nu W^\beta_\mu\right]\right\}^2\\&\quad+\left|D_\mu W^\beta_\nu-D_\nu W^\beta_\mu+ig\Gamma^\beta_{\mu\nu}\right|^2\ ,\end{aligned} \tag{28}$$

$$\Gamma^\beta_{\mu\nu}=\sum_{\alpha+\gamma=\beta}N_{\alpha,\gamma}\left[W^\alpha_\mu W^\gamma_\nu-W^\alpha_\nu W^\gamma_\mu\right]+\sum_{\alpha-\gamma=\beta}N_{\alpha,-\gamma}\left[W^\alpha_\mu W^{\gamma*}_\nu-W^\alpha_\nu W^{\gamma*}_\mu\right]\ , \tag{29}$$

where α,β and γ are positive roots, and $D_\mu W^\beta_\nu=\partial_\mu W^\beta_\nu-ig(\beta,h_k)A^k_\mu W^\beta_\nu$. A calculation of the field strength tensor is somewhat tedious but straightforward. The identities $\nabla_\mu(\alpha)n_\beta\equiv\partial_\mu n_\beta+ig[\alpha_\mu,n_\beta]=-ig(h_k,\beta)H_kn_\beta$ and $(\partial_\mu n_j,n_k)=(\partial_\mu n_k,n_j)=0$, which can be deduced from (27), are useful to simplify the computation. The Lagrangian density (28) is invariant under the Abelian gauge transformations

$$A^k_\mu\to A^k_\mu+\partial_\mu\xi_k\ ,\qquad W^\beta_\mu\to e^{ig(\beta,h_k)\xi_k}W^\beta_\mu\ . \tag{30}$$

The Abelian gauge group $[U(1)]^{N-1}$ is not a subgroup of the original gauge group. Just as in the SU(2) case, it is related to the fact that the basis elements n_β can be transformed locally

$$n_\beta \to e^{i(\beta,h_k)\xi_k} n_\beta \tag{31}$$

without spoiling both the orthogonality and commutation relations in the local Cartan-Weyl basis.

The gauge transformation law of the new variables can be found by the same method as in the SU(2) case. It depends on the choice of χ. There is a wide class of χ's, associated with Abelian projection gauges as explained in section 3, for which the gauge transformation law has a particularly simple form

$$\delta n_k = i[n_k, \varphi] \ , \quad \delta C_\mu^k = g^{-1}(n_k, \partial_\mu \varphi) \ , \quad \delta W_\mu = i[W_\mu, \varphi] \ . \tag{32}$$

In this case, the SU(N) gauge transformations are generated by adjoint transformations of any *rigid* local Cartan-Weyl basis, under which the field variables in the Abelian theory are invariant

$$\delta A_\mu^k = 0 \ , \quad \delta W_\mu^\beta = 0 \ . \tag{33}$$

All Abelian projections in which topological defects occur in Abelian components of projected connections fall into one class defined by the projection $n_k \to h_k$ in the new variables. A proof of the 't Hooft conjecture is a straightforward generalization of the SU(2) case. The key point is the gauge symmetry (32) which does not mix W_μ with the Abelian variables C_μ^k and n_k. Therefore the monopole dynamics in the SU(N) Yang-Mills theory is projection independent.

The coupling constants of the interaction of W_μ^β among each other and with the Maxwell fields A_μ^k in (28) are proportional to $N^{-1/2}$ because $N_{\beta,\gamma} \sim N^{-1/2}$ and $|(\beta, h_k)| \leq N^{-1/2}$, while $H_{\mu\nu}^k \sim N^{1/2}$. Therefore in the large N limit the dynamics of the topological fields n_k dominates [8].

Finally, we observe that precisely in *four* dimensional spacetime, the $3(N^2 - N)$ independent components of W_μ can be unified into a tensor $W_{\mu\nu}^{jk}$ which is *antisymmetric* in the Lorentz indices and *symmetric* in the Cartan indices, $W_{\mu\nu}^{jk} = -W_{\nu\mu}^{jk}$ and $W_{\mu\nu}^{jk} = W_{\mu\nu}^{kj}$. This suggests the following local parameterization to reveal a partial duality in the SU(N) Yang-Mills theory

$$W_\mu = \left\{ W_{\mu\nu}^{jk} + V_{\mu\nu}^{jk}(W, n) \right\} \alpha_\nu^{jk} \ , \quad \alpha_\mu^{jk} = i[\partial_\mu n_j, n_k] = \alpha_\mu^{kj} \ , \tag{34}$$

where the symmetric tensor $V_{\mu\nu}^{jk}$ can be specified as an explicit and local function of $W_{\mu\nu}^{jk}$ and n_k via a simple generalization of the method of section 5.

7. Conclusions. An Abelian structure and the Abelian magnetic symmetry can be established in the SU(N) Yang-Mills theory without any gauge fixing (or Abelian projections). By making use of such an Abelian theory we have shown that the effective dynamics of the topological degrees of freedom that are singled out as magnetic monopoles in Abelian projections is independent of the projection ('t Hooft conjecture). We have also generalized a parameterization of Faddeev and Niemi in order to study an off-shell dynamics of physical degrees of freedom which may form knot-like solitons in the infrared region of the Yang-Mills theory. A general parameterization of the Yang-Mills connection to reveal a partial duality between the topological field $\boldsymbol{n}$ and a dimensionless antisymmetric field $W_{\mu\nu}$ has been proposed. It is believed that the separation of the topological degrees of freedom of the Yang-Mills theory via a *change* of variables in the functional integral is important for developing the corresponding effective action by analytical means.

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